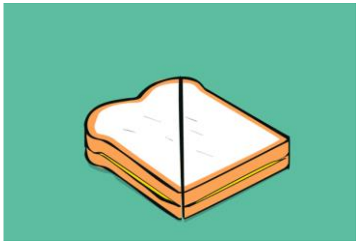
7. DIAGONALISATION

§7.1. Diagonal Matrices



Recall that a **diagonal matrix** is a square matrix where every component above or below the diagonal is zero. So A = (a_{ij}) is diagonal if $a_{ij} = 0$ whenever $i \neq j$. Some or all of the diagonal components can also be zero.

So a diagonal matrix has the form:

$$\begin{pmatrix} d_1 & 0 & \ldots\ldots & 0 \\ 0 & d_2 & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & \ldots\ldots & d_n \end{pmatrix}.$$

Sometimes we write this as **diag($\mathbf{d_1, d_2, \ldots, d_n}$)**.

Clearly the sum, difference and product of two $n \times n$ diagonal matrices are again diagonal.

The powers of a diagonal matrix are easy to compute.

If A = diag($d_1, d_2, \ldots, d_n$) then

$A^m = \text{diag}(d_1{}^m, d_2{}^m, \ldots, d_n{}^m)$.

Example 1: If $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ then

$$A^m = \begin{pmatrix} a_1{}^m & 0 & 0 \\ 0 & a_2{}^m & 0 \\ 0 & 0 & a_3{}^m \end{pmatrix}.$$

§7.2. Similar Matrices

Two matrices, A and B, are defined to be **similar** if there exists an invertible matrix S such that
$A = SBS^{-1}$.

Theorem 1: The relationship of being similar is an equivalence relation.

Proof: *Reflexive:* Let A be a square matrix.
Since $A = IAI^{-1}$, A is similar to itself.
Symmetric: Suppose that A is similar to B.
Then $A = SBS^{-1}$ for some invertible S.
Now $B = S^{-1}AS = (S^{-1})A(S^{-1})^{-1}$.
Since S^{-1} is invertible, B is similar to A.
Transitive:
Suppose A is similar to B and B is similar to C.
Then $A = SBS^{-1}$ and $B = TCT^{-1}$, for some invertible matrices S, T. Hence $A = S(TCT^{-1})S^{-1} = (ST)C(ST)^{-1}$. Since ST will be invertible, A is similar to C.

Similar matrices in the technical sense are similar in the informal sense. They have a lot of properties in common.

Theorem 2: Similar matrices have the same:

(1) Determinant;
(2) Trace;
(3) Characteristic polynomial;
(4) Eigenvectors.

Proof: We'll prove (3) first:
Suppose $A = SBS^{-1}$ for some invertible S.

$$
\begin{aligned}
\text{Then } \chi_A(\lambda) &= |\lambda I - A| \\
&= |\lambda I - SBS^{-1}| \\
&= |\lambda SIS^{-1} - SBS^{-1}| \\
&= |S(\lambda I - B)S^{-1}| \\
&= |S|.|\lambda I - B|.|S|^{-1} \\
&= |\lambda I - B| = \chi_B(\lambda).
\end{aligned}
$$

(4). Since similar matrices have the same characteristic polynomial, they must have the same eigenvalues.

(1) Since the determinant of a matrix is the product of the eigenvalues, similar matrices must have the same determinant.

(2) Since the trace of a matrix is the sum of the eigenvalues, similar matrices must have the same trace.

However similar matrices do not generally have the same eigenvectors.

Theorem 3: If $A = SBS^{-1}$ and $\mathbf{v}$ is an eigenvector of B, then $S\mathbf{v}$ is an eigenvector for A.
Proof: Suppose that $\mathbf{v}$ is an eigenvector for B.

Then $\mathbf{v} \neq \mathbf{0}$ and $B\mathbf{v} = \lambda\mathbf{v}$.
Hence $A(S\mathbf{v}) = (SBS^{-1})S\mathbf{v} = SB\mathbf{v} = S(\lambda\mathbf{v}) = \lambda(S\mathbf{v})$.
So $S\mathbf{v}$ is an eigenvector for B, corresponding to the same eigenvalue.
[Note that $S\mathbf{v} \neq \mathbf{0}$ since S is invertible and $\mathbf{v} \neq \mathbf{0}$.]

§7.3. Diagonalisable Matrices

A matrix, A, is defined to be **diagonalisable** if it is similar to a diagonal matrix, that is, if there exists an invertible matrix S such that $A = SDS^{-1}$. If it is not diagonalisable we say that A is **non-diagonalisable**.

If the eigenvalues of the $n \times n$ diagonalisable matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then A must be similar to the diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The useful property of diagonalisable matrices is that their powers are easily calculated.

Theorem 4: If $A = SDS^{-1}$, where S is invertible and D is diagonal then $A^m = SD^mS^{-1}$.

Proof: $A^m = (SDS^{-1})(SDS^{-1}) \ldots (SDS^{-1})(SDS^{-1})$ (m groups)

$$= SD(S^{-1}S)D(S^{-1}S) \ldots (S^{-1}S)D(S^{-1}S)DS^{-1}$$
$$= SD^mS^{-1}.$$

Corollary: If $A = SDS^{-1}$ is a diagonalisable matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then for any integer, m,
$A^m = S.\text{diag}(\lambda_1{}^m, \lambda_2{}^m, \ldots, \lambda_n{}^m).S$

A natural question is "Are *all* matrices diagonalisable?" The answer is "No, but they generally are."

Example 2: The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is non-diagonalisable.

Proof: It has characteristic polynomial
$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$.
Hence it has a double eigenvalue of 1.
If A was similar to a diagonal matrix D then we would have to have D = diag(1, 1) = I.
But if $A = SDS^{-1}$ then A = I, which is clearly not so.

Here's a partial list of diagonalisable matrices. It doesn't cover them all, but it will be clear that to be diagonalisable is the rule, rather than not the exception. We shall show the following.

DIAGONALISABLE MATRICES INCLUDE:
(1) $n \times n$ matrices with n distinct eigenvalues;
(2) matrices A such that $A^m = I$ for some positive integer m;
(3) real symmetric matrices.

If A, S are $n \times n$ matrices and the columns of S are eigenvectors for A then S is called an **eigenmatrix** of A.

Theorem 5: If $S = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is an eigenmatrix for A then AS = SD where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and λ_i is the eigenvalue corresponding to the eigenvector $\mathbf{v}_i$.

Proof:
$$\begin{aligned} AS &= A(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \\ &= (A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n) \\ &= (\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \ldots, \lambda_n\mathbf{v}_n) \\ &= (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} = SD. \end{aligned}$$

Corollary 1: If S is an invertible eigenmatrix for A then $A = SDS^{-1}$.

Corollary 2: A matrix is diagonalisable if and only if it has an invertible eigenmatrix.

Proof: Suppose A has an invertible eigenmatrix S. Then AS = SD, as above, and since S is invertible we may write this as $A = SDS^{-1}$.

Conversely suppose that A is diagonalisable.

Then $A = SDS^{-1}$ for some diagonal matrix D and invertible matrix S. Then AS = SD.

Let $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$ and $S = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.

Hence $AS = (A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n) = (\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \ldots, \lambda_n\mathbf{v}_n)$ and so each $\mathbf{v}_i$ is an eigenvector for A.

Example 3: Diagonalise the matrix A = $\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$.

Solution: tr(A) = 5 and $|A| = 6$.
So $\chi_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$.

λ = 2: A – 2I = $\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$

so a corresponding eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

λ = 3: A – 3I = $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$

so a corresponding eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Let S = $\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and D = $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Then $S^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$.

We can check that $A = SDS^{-1}$.

Example 4: Diagonalise the matrix

$$A = \begin{pmatrix} 7 & 140 & -240 \\ 6 & 65 & -120 \\ 4 & 42 & -78 \end{pmatrix}.$$

Solution: $\mathrm{tr}_1(A) = \mathrm{tr}(A) = 7 + 65 - 78 = -6$.

$$\mathrm{tr}_2(A) = \begin{vmatrix} 7 & 140 \\ 6 & 65 \end{vmatrix} + \begin{vmatrix} 7 & -240 \\ 4 & -78 \end{vmatrix} + \begin{vmatrix} 65 & -120 \\ 42 & -78 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 75 \\ 6 & 65 \end{vmatrix} + \begin{vmatrix} -1 & -84 \\ 4 & -78 \end{vmatrix} + \begin{vmatrix} 23 & -42 \\ 42 & -78 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 75 \\ 6 & 65 \end{vmatrix} + \begin{vmatrix} -1 & -84 \\ 4 & -78 \end{vmatrix} + \begin{vmatrix} 23 & -42 \\ -4 & 6 \end{vmatrix} \quad R_3 - 2R_1$$

$$= (65 - 450) + (78 + 336) + (138 - 168)$$

$$= -1$$

$$tr_3(A) = |A| = \begin{vmatrix} 7 & 140 & -240 \\ 6 & 65 & -120 \\ 4 & 42 & -78 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 75 & -120 \\ 6 & 65 & -120 \\ 4 & 42 & -78 \end{vmatrix} \quad R_1 - R_2$$

$$= \begin{vmatrix} 1 & 75 & -120 \\ 0 & -385 & 600 \\ 0 & -258 & 402 \end{vmatrix} \quad R_2 - 6R_1,\ R_3 - 4R_1$$

$$= \begin{vmatrix} -385 & 600 \\ -258 & 402 \end{vmatrix}$$

$$= 30.$$

So $\chi_A(\lambda) = \lambda^3 + 6\lambda^2 - \lambda - 30$.
By inspection $\lambda = 2$ is a zero.
So $\chi_A(\lambda) = (\lambda - 2)(\lambda^2 + 8\lambda + 15)$
$= (\lambda - 2)(\lambda + 3)(\lambda + 5)$.

Hence the eigenvalues are $\lambda = 2, -3, -5$.

λ = 2:

$$A - \lambda I = \begin{pmatrix} 5 & 140 & -240 \\ 6 & 63 & -120 \\ 4 & 42 & -80 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -77 & 120 \\ 5 & 140 & -240 \\ 4 & 42 & -80 \end{pmatrix}$$

$R_2 - R_1,\ R_1 \leftrightarrow R_2$

$$\rightarrow \begin{pmatrix} 1 & -77 & 120 \\ 0 & 525 & -840 \\ 0 & 350 & -560 \end{pmatrix} \quad R_2 - 5R_1,\ R_3 - 4R_1$$

$$\rightarrow \begin{pmatrix} 1 & -77 & 120 \\ 0 & 5 & -8 \\ 0 & 350 & -560 \end{pmatrix} \quad R_2 \div 105$$

$$\rightarrow \begin{pmatrix} 1 & -77 & 120 \\ 0 & 5 & -8 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 - 70R_2$$

Put z = 5k, 5y = 8(5k) so y = 8k and
$x = 77(8k) - 120(5k) = 16k$.

Note that putting z = 5k avoids fractions. We could jave put z = k and then multiplied the eigenvector we obtained by 5. Remember any non-zero scalar multiple of an eigenvector is an equivalent eigenvector.

So $\begin{pmatrix} 16 \\ 8 \\ 5 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

λ = –3:

$$A - \lambda I = \begin{pmatrix} 10 & 140 & -240 \\ 6 & 68 & -120 \\ 4 & 42 & -75 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 14 & -24 \\ 6 & 68 & -120 \\ 4 & 42 & -75 \end{pmatrix}$$

$R_2 \div 10$

$$\rightarrow \begin{pmatrix} 1 & 14 & -24 \\ 0 & -16 & 24 \\ 0 & -14 & 21 \end{pmatrix} \quad R_2 - 6R_1,\ R_3 - 4R_1$$

$$\rightarrow \begin{pmatrix} 1 & 14 & -24 \\ 0 & 2 & -3 \\ 0 & -14 & 21 \end{pmatrix} \quad R_2 \div (-8)$$

$$\rightarrow \begin{pmatrix} 1 & 14 & -24 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 + 7R_2$$

Put z = $2k$, $2y = 3(2k)$ so $y = 3k$ and
$x = -14(3k) + 24(2k) = 6k$.

So $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue –3.

$\lambda = -5$:

$$A - \lambda I = \begin{pmatrix} 12 & 140 & -240 \\ 6 & 70 & -120 \\ 4 & 42 & -73 \end{pmatrix} \to \begin{pmatrix} 3 & 35 & -60 \\ 6 & 70 & -120 \\ 4 & 42 & -73 \end{pmatrix}$$

$R_1 \div 4$

$$\to \begin{pmatrix} 1 & 7 & -13 \\ 6 & 70 & -120 \\ 3 & 35 & -60 \end{pmatrix} \quad R_3 - R_1,\ R_1 \leftrightarrow R_3$$

$$\to \begin{pmatrix} 1 & 7 & -13 \\ 0 & 28 & -42 \\ 0 & 14 & -21 \end{pmatrix} \quad R_2 - 6R_1,\ R_3 - 3R_1$$

$$\to \begin{pmatrix} 1 & 7 & -13 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad R_2 \div 14,\ R_3 - 7R_2$$

Put $z = 2k$, $2y = 3(2k)$ so $y = 3k$ and
$x = -7(3k) + 13(2k) = 5k$.

So $\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalues –5.

Take S = $\begin{pmatrix} 16 & 6 & 5 \\ 8 & 3 & 3 \\ 5 & 2 & 2 \end{pmatrix}$ and D = $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

Then A = SDS^{-1}. We can check this by performing the appropriate multiplication.

Example 5: Show that A = $\begin{pmatrix} 3 & -22 & 18 \\ 3 & -14 & 9 \\ 2 & -8 & 4 \end{pmatrix}$ is not diagonalisable.

Solution: We begin by working out the characteristic polynomial.

tr(A) = 3 – 14 + 4 = –7.

$$tr_2(A) = \begin{vmatrix} 3 & -22 \\ 3 & -14 \end{vmatrix} + \begin{vmatrix} 3 & 18 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} -14 & 9 \\ -8 & 4 \end{vmatrix}$$

$$= -42 + 66 + 12 - 36 - 56 + 72$$

$$= 16.$$

$$|A| = \begin{vmatrix} 3 & -22 & 18 \\ 3 & -14 & 9 \\ 2 & -8 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -22 & 18 \\ 0 & 8 & -9 \\ 2 & -8 & 4 \end{vmatrix}$$

$= 3(32 – 72) + 2(198 – 144)$
$= – 120 + 108$
$= – 12.$

Hence $\chi_A(\lambda) = \lambda^3 + 7\lambda^2 + 16\lambda + 12$
$= (\lambda + 2)(\lambda + 2)(\lambda + 3).$

[We factorise this by trying certain integer values until we discover that $\lambda = –2$ is a zero. Hence $\lambda + 2$ is a factor. We then divide $\chi_A(\lambda)$ by $\lambda + 2$ and solve the resulting quadratic.]

So the eigenvalues are $\lambda = –2$ (twice) and $\lambda = –3$.

$$\boldsymbol{\lambda = -3:}\ A + 3I = \begin{pmatrix} 6 & -22 & 18 \\ 3 & -11 & 9 \\ 2 & -8 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 2 & -8 & 7 \\ 6 & -22 & 18 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 3 \\ 0 & -4 & 6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ is an eigenvector.

$$\boldsymbol{\lambda = -2}\textbf{:}\ A + 2I = \begin{pmatrix} 5 & -22 & 18 \\ 3 & -12 & 9 \\ 2 & -8 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & 3 \\ 2 & -8 & 6 \\ 5 & -22 & 18 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -4 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ is an eigenvector.

If A was diagonalisable there would have been two eigenvectors for $\lambda = -2$ that are not multiples of one another.

The steps involved in diagonalisation are as follows.

DIAGONALISATION OF AN n × n MATRIX A

(1) Compute $\mathrm{tr}(A)$, $\mathrm{tr}_2(A)$, ... $\mathrm{tr}_{n-1}(A)$ and $|A|$.

(2) Write down $\chi(\lambda) = \lambda^n - \mathrm{tr}(A)\lambda^{n-1} + \mathrm{tr}_2(A)\lambda^{n-2}$

$$- \ldots + (-1)^{n-1}\mathrm{tr}_{n-1}(A) + (-1)^n|A|.$$

(3) Solve $\chi(\lambda) = 0$ to find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
(4) For each eigenvalue find an eigenvector.
(5) If you can get an invertible eigenmatrix S then A is diagonalisable. Otherwise it is not diagonalisable.
(6) Let $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
(7) Find S^{-1} and check that $A = SDS^{-1}$.

NOTE: The columns of S can be in any order so long as the components of D are in the same order.

§7.4. Powers of Matrices

Computing integer powers of a matrix is just a matter of applying matrix multiplication to the matrix or its inverse. But if n is large this can involve an enormous amount of work. And often we require a formula for A^n in terms of *n*. Sometimes we're interested in the behaviour of A^n as n approaches infinity.

Example 6: If $A = \begin{pmatrix} 7 & 140 & -240 \\ 6 & 65 & -120 \\ 4 & 42 & -78 \end{pmatrix}$, find A^5 and A^{-4}.

Solution: In example 1 we showed that A is diagonalisable, and that $A = SDS^{-1}$ where

$$S = \begin{pmatrix} 16 & 6 & 5 \\ 8 & 3 & 3 \\ 5 & 2 & 2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

We must first find S^{-1}. Since S is only a 3×3 matrix it is just as easy to use the cofactor method.

$$\text{adj}(S) = \begin{pmatrix} 0 & -1 & 1 \\ -2 & 7 & -2 \\ 3 & -8 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -2 & 3 \\ -1 & 7 & -8 \\ 1 & -2 & 0 \end{pmatrix} \text{ and}$$

$$|S| = \begin{vmatrix} 16 & 6 & 5 \\ 8 & 3 & 3 \\ 5 & 2 & 2 \end{vmatrix} = 16(0) - 6(1) + 5(1) = -1.$$

Hence $S^{-1} = \begin{pmatrix} 0 & 2 & -3 \\ 1 & -7 & 8 \\ -1 & 2 & 0 \end{pmatrix}$.

So $A = SDS^{-1}$ and so $A^5 = SD^5S^{-1}$ =

$$\begin{pmatrix} 16 & 6 & 5 \\ 8 & 3 & 3 \\ 5 & 2 & 2 \end{pmatrix} \begin{pmatrix} 32 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & -3125 \end{pmatrix} \begin{pmatrix} 0 & 2 & -3 \\ 1 & -7 & 8 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 512 & -1458 & -15625 \\ 256 & -729 & -9375 \\ 160 & -486 & -6250 \end{pmatrix} \begin{pmatrix} 0 & 2 & -3 \\ 1 & -7 & 8 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 14167 & -20020 & -13200 \\ 8646 & -13135 & -6600 \\ 5764 & -8778 & -4368 \end{pmatrix}.$$

To compute A^{-4} we could find the inverse of A and then raise it to the 4th power, but it's simpler to do it directly. This time, instead of computing it exactly we'll approximate each component to 4 decimal places.

$$A^{-4} = SD^{-4}S^{-1} =$$

$$\begin{pmatrix} 16 & 6 & 5 \\ 8 & 3 & 3 \\ 5 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0.0625 & 0 & 0 \\ 0 & 0.0123 & 0 \\ 0 & 0 & 0.0016 \end{pmatrix} \begin{pmatrix} 0 & 2 & -3 \\ 1 & -7 & 8 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0.0738 & 0.008 \\ 0.5 & 0.0369 & 0.0048 \\ 0.3125 & 0.0246 & 0.0032 \end{pmatrix} \begin{pmatrix} 0 & 2 & -3 \\ 1 & -7 & 8 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.073 & 1.4994 & -2.4096 \\ 0.0321 & 0.7513 & -1.2048 \\ 0.0214 & 0.4592 & -0.7407 \end{pmatrix}.$$

Suppose A(x) is a matrix where each component is a function of x. We define the **limit** of A(x) as x approaches a real number or $\pm\infty$ to be the matrix whose components are the respective limits, that is, if these limits exist.

Example 7: (i) $\lim\limits_{x \to 0} \begin{pmatrix} \dfrac{\sin x}{x} & x \\ \tan x & 2\cos x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$,

(ii) $\lim\limits_{x \to 0} \begin{pmatrix} 1 - \dfrac{1}{x} & 2 + e^{-x} \\ 3 - \dfrac{1}{x^2} & \left(\dfrac{1}{2}\right)^x \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$.

Since the limit of a sum, difference and product of functions is the sum, difference and product of the limits, and matrix addition, subtraction and multiplication can be expressed in terms of these operations on the components, the limit of a sum, difference and product of matrices is the sum, difference and product of the respective matrices.

If each λ_i satisfies $-1 < \lambda_i \le 1$ then $\begin{pmatrix} \lambda_1{}^m & 0 & \dots & 0 \\ 0 & \lambda_2{}^m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n{}^m \end{pmatrix}$ converges as $m \to \infty$ and the limit is a diagonal matrix of 1's and 0's where the 1's on the diagonal correspond to the λ_i that are equal to 1.

Example 8: $\begin{pmatrix} 0.9^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-0.3)^n \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

as $n \to \infty$.

If a diagonalisable $n \times n$ matrix A has all its eigenvalues in the interval $(-1, 0]$ then A^n converges to a limit as $n \to \infty$.

Example 9:

(i) Show that if $A = \begin{pmatrix} -1.5 & 0.5 \\ -5 & 2 \end{pmatrix}$ then A^n converges as n approaches infinity.

(ii) Find the limit.

Solution: (i) $|\lambda I - A| = \begin{vmatrix} \lambda+1.5 & -0.5 \\ 5 & \lambda-2 \end{vmatrix}$

$$= (\lambda + 1.5)(\lambda - 2) + 2.5$$
$$= \lambda^2 - 0.5\lambda - 0.5$$
$$= (\lambda - 1)(\lambda + 0.5).$$

So the eigenvalues of A are 1, – 0.5.

Since these are in the interval (–1, 1] the matrix A^n converges as $n \to \infty$.

(ii) To find the limit we must diagonalise.

λ = 1: $A - I = \begin{pmatrix} -2.5 & 0.5 \\ -5 & 1 \end{pmatrix} \to \begin{pmatrix} -5 & 1 \\ 0 & 0 \end{pmatrix}$

so $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ is an eigenvector for $\lambda = 1$.

λ = – 0.5: $A + 0.5I = \begin{pmatrix} -1 & 0.5 \\ -5 & 2.5 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$

so $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $\lambda = -0.5$.

Let $S = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}$.

Then $A = SDS^{-1}$ and $A^n = SD^nS^{-1}$.

Since $D^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as $n \to \infty$, $A^n \to S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1}$.

Now $S^{-1} = -\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -5 & 1 \end{pmatrix}$ so

$$\lim_{n\to\infty} A^n = -\frac{1}{3}\begin{pmatrix}1 & 1\\ 5 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}2 & -1\\ -5 & 1\end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix}1 & 0\\ 5 & 0\end{pmatrix}\begin{pmatrix}2 & -1\\ -5 & 1\end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix}2 & -1\\ 10 & -5\end{pmatrix}$$

$$= \begin{pmatrix}-2/3 & 1/3\\ -10/3 & 5/3\end{pmatrix}.$$

Fractional powers can be found for a diagonalisable matrix in the same way.

Example 10: If A = $\begin{pmatrix}4 & 2\\ -1 & 1\end{pmatrix}$, find a square root of A, if one exists.

Solution: If S = $\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}$ and D = $\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$ then

A = SDS^{-1} [We omit the details.]

Let B = $S\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{3}\end{pmatrix} S^{-1}$.

Now $S^{-1} = \begin{pmatrix}-1 & -2\\ 1 & 1\end{pmatrix}$.

$$\therefore\ B = \begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{3}\end{pmatrix}\begin{pmatrix}-1 & -2\\ 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & 2\sqrt{3} \\ -\sqrt{2} & -\sqrt{3} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{3}-\sqrt{2} & 2\sqrt{3}-2\sqrt{2} \\ \sqrt{2}-\sqrt{3} & 2\sqrt{2}-\sqrt{3} \end{pmatrix}.$$

In fact an $n \times n$ diagonalisable matrix has 2^n square roots (if we allow non-real components) since there are two square roots for each eigenvalue.

In this example there are 4 square roots of A. These are obtained by replacing √2 by ±√2 and √3 by ±√3. Thus the four square roots of A are

$$\pm \begin{pmatrix} 2\sqrt{3}-\sqrt{2} & 2\sqrt{3}-2\sqrt{2} \\ \sqrt{2}-\sqrt{3} & 2\sqrt{2}-\sqrt{3} \end{pmatrix} \text{ and } \pm \begin{pmatrix} 2\sqrt{3}+\sqrt{2} & 2\sqrt{3}+2\sqrt{2} \\ -\sqrt{2}-\sqrt{3} & -2\sqrt{2}-\sqrt{3} \end{pmatrix}$$

An $n \times n$ non-diagonalisable matrix can have as few as 2 square roots, or even infinitely many. For example, in Chapter 2 we saw that the square roots of the identity matrix are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix} \text{ for all } k$$

and $\begin{pmatrix} a & \dfrac{1-a^2}{c} \\ c & -a \end{pmatrix}$ for all a, and all $c \neq 0$

Example 12: Find the square roots of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution:

Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and suppose that $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $a^2 + bc = d^2 + bc = b(a + d) = 1$ and $c(a + d) = 0$.

Hence $a + d \neq 0$ and so $c = 0$.

Since $a^2 = d^2 = 1$ we can only have:

$$a = d = 1 \text{ or } a = d = -1.$$

In the first case $b = ½$ and in the second, $b = -½$.

So $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ only has 2 square roots:

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \text{and} \begin{pmatrix} -1 & -1/2 \\ 0 & -1 \end{pmatrix}$$

§7.5. Recurrence Equations

A **sequence** is an ordered list of numbers or functions a_1, a_2, ... (Sometimes the list starts with a_0 instead of a_1.) Although a sequence can be completely random it only becomes interesting mathematically if there is some rule or pattern that describes the list. One type of rule is where we have a formula for the n'th term. Another type of rule is where each term is defined in terms of the previous term, or a certain number of previous terms. Such a rule is called a **recurrence equation**.

Example 13: $a_n = n^2$ describes the sequence:

$$1, 4, 9, 16, ...$$

$a_n = 3a_{n-1}$ describes the sequence 1, 3, 9, 27, ... It also describes the sequence 2, 6, 18, 54, ...

As the above example shows a recurrence equation doesn't define the sequence uniquely. We must be given one or more terms explicitly until the recurrence equation can start.

Example 14: There is a famous sequence, called the Fibonacci sequence, that is described as follows:

$$a_{n+2} = a_{n+1} + a_n;$$
$$a_0 = 1;$$
$$a_1 = 1.$$

The sequence begins as follows:

$$1, 1, 2, 3, 5, 8, 13, ...$$

Now it's preferable to have an explicit formula for the n'th term than to have a recurrence equation. If $a_n = n^2$ it's easy to see that a_{1000} is 1,000,000. To find the 1000^{th} term of the Fibonacci sequence we appear to have to compute the sequence all the way up to u_{1000}.

Also, if we have an explicit formula, we can make general statements about the sequence, such as whether it converges.

If we have the sequence defined by:

$$a_n = 3a_{n-1},\ a_1 = 2,$$

we can obtain an explicit formula for a_n. This is a Geometric Progression with first term 2 and common ratio of 3. The n'th term is $a_n = 2.3^{n-1}$.

A **linear recurrence** is one of the form:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \ldots + c_k a_n$$

where the c_i's are constants. The **order** of this recurrence is k.

For example the Fibonacci sequence is a second order linear recurrence. Using matrices it's possible to find an explicit formula for a_n in terms of n in such cases. The technique involves considering a more general situation.

Example 15: Suppose a_1, a_2, ... and b_1, b_2, ... are two sequences where

$$a_{n+1} = 4a_n + 2b_n;$$
$$b_{n+1} = -a_n + b_n;$$
$$a_0 = 4,\ b_0 = 7.$$

Find explicit expressions for a_n and b_n in terms of n.

Solution:

We need to generate both sequence together.

The first sequence is 4, 30, 126, 450, ...

The second sequence is 7, 3, –27, –153, ...

But suppose we define $\mathbf{v_n} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$. Instead of two sequences of numbers we have a single sequence of vectors. Moreover the recurrence equations can be expressed as a single matrix/vector equation:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

If we put $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$ then $\mathbf{v}_{n+1} = A\mathbf{v}_n$.

Remarkably the pair of intertwined sequences becomes, not only a single sequence, but a very simple one at that. It's simply a geometric sequence with common ratio A.

It doesn't matter that A is a matrix and not just a number. The formula for the *n*'th term still works:

$$\mathbf{v}_n = A^n \mathbf{v}_0.$$

Here $\mathbf{v_0} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$. All we need to do is to diagonalise the matrix and hence find A^n in terms of n.

We can diagonalise A and get $A = SDS^{-1}$ where

$$S = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix},\ D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{and } S^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}.$$

So $A^n\mathbf{v}_0 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix}$.

$$= \begin{pmatrix} 2^n & 2.3^n \\ -2^n & -3^n \end{pmatrix} \begin{pmatrix} -18 \\ 11 \end{pmatrix}$$

$$= \begin{pmatrix} -18.2^n + 22.3^n \\ 18.2^n - 11.3^n \end{pmatrix}.$$

Hence $a_n = 22.\ 3^n - 18.\ 2^n$ and

$b_n = 18.\ 2^n - 11.\ 3^n$.

To solve a higher order recurrence:

$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + ... + a_{k-1} u_{n+1} + a_k u_n$

we turn it into a first order system by defining

$$\mathbf{v}_n = \begin{pmatrix} u_{n+k-1} \\ u_{n+k-2} \\ \ldots.. \\ u_n \end{pmatrix}. \text{ Then } \mathbf{v}_{n+k} = \begin{pmatrix} a_1 & a_2 & \ldots a_{k-1} & a_k \\ 1 & 0 \ldots & 0 & 0 \\ \ldots & \ldots\ldots\ldots & \ldots & \ldots \\ 0 & 0 \ldots & 1 & 0 \end{pmatrix} \mathbf{v}_n.$$

Theorem 4: If the $k \times k$ matrix A is diagonalisable with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$, the components of A^n and of $A^n\mathbf{v}$ for any $\mathbf{v}$, have the form $C_1\lambda_1{}^n + C_2\lambda_2{}^n + ... + C_k\lambda_k{}^n$ for constants $C_1, C_2, ..., C_k$.

Example 16: Solve the recurrence:

$$\begin{cases} u_{n+3} = 6u_{n+2} - 11u_{n+1} + 6u_n \\ u_0 = 1 \\ u_1 = 2 \\ u_2 = 3 \end{cases}.$$

Solution: Let R = $\begin{pmatrix} 6 & -4 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

tr(R) = 6, tr_2(R) = 11 and |R| = 6.

Hence $\chi_R(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$

$= (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

So the eigenvalues are λ = 1, 2, 3.

Hence $u_n = A + B2^n + C3^n$ for some A, B, C.

We can substitute n = 1, 2, 3 and get 3 equations in A, B, C which we can solve in the usual way/

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & 9 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right).$$

Hence $C = -\frac{1}{2}$, B = 2, $A = -\frac{1}{2}$.

So $u_n = \dfrac{-1 + 2^{n+2} - 3^n}{2}$ for all n.

§7.6. Probability Matrices

The **probability** of an event is a number between 0 and 1, inclusive, where a zero probability means that it cannot happen, while the probability of a certain event is 1. If we have a number of independent events we can interpret the probability as a proportion.

Example 17: The probability of a die (singular of dice) coming up 4 is 1 in 6, or $\frac{1}{6}$, because there are 6 equally likely possibilities. If we throw 60 dice the expected proportion of dice that show 6 is $\frac{1}{6}$ and the expected number is $\frac{60}{6}$, or 10.

Suppose we have two related events, E and F, and their probabilities are P€ and P(F). Suppose that the probability of E occurring, given that F has occurred, is P(E|F). This is called a **conditional probability**.

$$\text{Then } P(E) = P(E|F).P(F).$$

Suppose that there is a third event, G, such that F and G can't both occur, with probability P(G).

$$\text{Then } P(E) = P(E|F)P(F) + P(E|G)P(G).$$

Example 18: Suppose that the probability of rain is 0.3. Suppose that the probability of the Spring Fair being cancelled if it rains is 0.6 and the probability of it being cancelled if it doesn't rain (that is for some other reason) is 0.1. Then the probability of the fair being cancelled at all will be: $0.3 \times 0.6 + 0.7 \times 0.1 = 0.18 + 0.07 = 0.25$.

Suppose a system has n possible states S_1, S_2, ..., S_n and at any instant it is in one of these states. Let the probability of moving from state S_i to state S_j in one unit of time be P_{ij}. If the probability of the system being in state S_i at time t is $P_i^{(t)}$, then:

$$P_i^{(t+1)} = P_1^{(t)}P_{1i} + P_2^{(t)}P_{2i} + \ldots + P_n^{(t)}P_{ni}.$$

Let's use vectors and matrices to study this system. Let $\mathbf{v}^{(t)} = \begin{pmatrix} P_1^{(t)} \\ P_2^{(t)} \\ \ldots \\ P_n^{(t)} \end{pmatrix}$ and let $P = (P_{ij})$.

Then $\mathbf{v}^{(t+1)} = P^T\mathbf{v}^{(t)}$. Note the need to transpose P, because P_{1i}, P_{2i} … are the components in the i'th column of P not the i'th row.

The matrix P is called the **transition matrix** of the system. Note that if $S_1, S_2, \ldots, S_n$ is a complete list of all the states then each row of P must total 1, since if the system is currently in state i it must be in state j, for some j after one unit of time

Example 17: The system might be the age structure of the population with S_r being the state of being aged r years. We would take one year as the unit of time and P_{rs} would be the probability of someone r years being s years old (and hence still alive) one year later. If it wasn't for the problem of death this transition matrix would be the infinite matrix:

.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Note that here we are ignoring birth (or migration). Probably we only need to go up to state S_{100}, which can include all those 100 or over, and a state S_{101} for 'dead'. But because of mortality at all ages, especially infant mortality, the 1's just above the diagonals will be slightly reduced.

The following is a typical transition matrix for the age structure of a population, expressed as a table, rather than as a matrix, to simplify layout.

AGE	**0**	**1**	**2**	**3**	**.....**	**99**	**≥ 100**	**dead**
0	0	0.95	0	0	.....	0	0	0.05
1	0	0	0.99	0	.....	0	0	0.01
2	0	0	0	0.997	.....	0	0	0.003
	.....	.....	.....	.....	.....	.....	.....	.....
99	0	0	0	0	.....	0	0.66	0.34
≥ 100	0	0	0	0	.....		0.5	0.5
dead	0	0	0	0	.....		0	1

Here the first row and column correspond to state S_0, those in their first year of life. The first row reflects the fact that 5% of those in their first year of life fail to reach their first birthday. (This is obviously a third world country!)

The first column of all zeros reflects the fact that nobody, whatever their age, will be in their first year of life next year. Here we are taking a fixed cohort of people and are not considering births.

The last row and column represent state S_{101}, the state of being dead. Clearly all of these remain dead after a further year. You can see that the death rate drops to 1% for those aged 1 and even further to 0.3% for those aged 2.

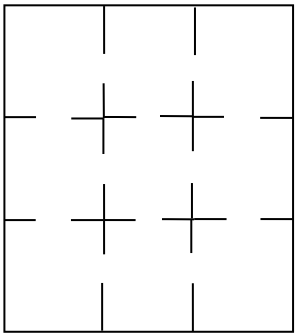
Of those aged 99 this year, 34% will die within the year with the remaining 66% moving into the '100 and over' group.

If $P = (P_{ij})$ is an $n \times n$ probability matrix and the vector $\mathbf{v}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \cdots \\ x_n^{(k)} \end{pmatrix}$ is the vector giving the numbers in each state at time k then the expected number in state j after one year is $P_{1j}x_1^{(k)} + P_{2j}x_2^{(k)} + \ldots + P_{nj}x_j^{(k)}$,
which is the j'th component of $P\mathbf{v}^{(k)}$.

Hence $\mathbf{v}^{(k+1)} = \mathrm{P}\mathbf{v}^{(k)}$ and so $\mathbf{v}^{(k)} = \mathrm{P}^k\mathbf{v}^{(0)}$. This shows the need to be able to find a formula for the k'th power of a matrix.

A matrix of size 102×102 as in the above example would require a computer to analyse. The following example has only three states and lends itself to manual calculation.

Example 18: A robot vacuum cleaner moves about the floor randomly, picking up dust. If it reaches a doorway it will go through into the next room. Suppose we have 9 rooms as follows.



For each door let p be the probability that the robot will move through that door in a 10 minute interval. We assume that the probability that it will move through a second door in that time interval to be negligible.

We have 9 rooms and could therefore potentially have to deal with a 9×9 matrix. However by symmetry we can combine the four corner rooms into one state and the four edge rooms also into one state.

Let S_1 be the state of being in one of the corner rooms, S_2 the state of being in one of the edge rooms and S_3 the state of being in the middle room. The transition matrix is

$$P = \begin{array}{c} \begin{pmatrix} 1-2p & 2p & 0 \\ 2p & 1-3p & p \\ 0 & 4p & 1-4p \end{pmatrix} \end{array} \begin{array}{l} \text{corner} \\ \text{edge} \\ \text{middle} \end{array}$$

Let's suppose that $p = 0.1$.

Then $P = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0 & 0.4 & 0.6 \end{pmatrix}$.

Let $Q = P^T = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.7 & 0.4 \\ 0 & 0.1 & 0.6 \end{pmatrix}$

If we start with such a robot vacuum cleaner in a corner room we have $\mathbf{v}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and so the expected distribution after 10 minutes is

$$P^T\mathbf{v}^{(0)} = Q\mathbf{v}^{(0)} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.7 & 0.4 \\ 0 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}.$$

After 20 minutes the distribution is:

$$Q^2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0.8 & 0.2 & 0\\0.2 & 0.7 & 0.4\\0 & 0.1 & 0.6\end{pmatrix}\begin{pmatrix}0.8\\0.2\\0\end{pmatrix} = \begin{pmatrix}0.68\\0.30\\0.02\end{pmatrix}$$

To determine the distribution after 24 hours would require a very long calculation. Fortunately we can find an explicit formula for Q^{144} or any power of Q for that matter.

First we find the characteristic polynomial.
$tr_1(Q) = 2.1$,
$tr_2(Q) = 1.38$,
$tr_3(Q) = 0.28$.
So $\chi_Q(\lambda) = \lambda^3 - 2.1\lambda^2 + 1.38\lambda - 0.28$.

Solving a cubic can be tricky, but fortunately we know that one of the eigenvalues is 1. How do we know this? Because the rows of P total 1 and so $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector for P with corresponding eigenvalue 1. This means that $Q = P^T$ will also have 1 as an eigenvalue (although it will have a different corresponding eigenvector).
Dividing $\chi_Q(\lambda)$ by $\lambda - 1$ gives $\lambda^2 - 1.1\lambda + 0.28$.
Solving by the quadratic formula we get zeros of 0.7 and 0.4.
So the eigenvalues of Q are 1, 0.7 and 0.4.

$$\boldsymbol{\lambda = 1:}\ Q - I = \begin{pmatrix} -0.2 & 0.2 & 0 \\ 0.2 & -0.3 & 0.4 \\ 0 & 0.1 & -0.4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0.2 & -0.3 & 0.4 \\ 0 & 0.1 & -0.4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -0.1 & 0.4 \\ 0 & 0.1 & -0.4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 1$.

$$\boldsymbol{\lambda = 0.7:}\ Q - 0.7I = \begin{pmatrix} 0.1 & 0.2 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & -0.1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & -0.1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & -0.4 & 0.4 \\ 0 & 0.1 & -0.1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 0.7$.

λ = 0.4: $Q - 0.4I = \begin{pmatrix} 0.4 & 0.2 & 0 \\ 0.2 & 0.3 & 0.4 \\ 0 & 0.1 & 0.2 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & 0.5 & 0 \\ 0.2 & 0.3 & 0.4 \\ 0 & 0.1 & 0.2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0.2 & 0.4 \\ 0 & 0.1 & 0.2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 0.4$.

Let $S = \begin{pmatrix} 4 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = .\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}$

We now must calculate S^{-1}.

$S^{-1} = \dfrac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 4 \\ 1 & -2 & 4 \end{pmatrix}$ (I have omitted the details.)

Then $\mathbf{v}^{(n)} = Q^n\mathbf{v}^{(0)} = SD^nS^{-1}\mathbf{v}^{(0)}$

$$= \frac{1}{9}\begin{pmatrix} 4 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7^n & 0 \\ 0 & 0 & 0.4^n \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 4 \\ 1 & -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7^n & 0 \\ 0 & 0 & 0.4^n \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \times 0.7^n \\ 0.4^n \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 + 4 \times 0.7^n + 0.4^n \\ 4 - 2 \times 0.7^n - 2 \times 0.4^n \\ 1 - 2 \times 0.7^n + 0.4^n \end{pmatrix}$$

Clearly when n is large 0.7^n and 0.4^n are negligible.
For example 0.7^{144} is about 5×10^{-23}.

So for large n, $\mathbf{v}^{(n)} \approx \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$.

This means that the probability of the robot vacuum cleaner being in a corner room is $\frac{4}{9}$. Since there are 4 such rooms the probability of it being in any one of those rooms is $\frac{1}{9}$. In fact, this is true of any of the 9 rooms, a fact that we might have expected. However there might have been some factors that make a transition through some doors more or less than through others. For example there might be double doors, wider than normal ones. Of course here we couldn't treat all corner rooms the same and we would probably have had to use a 9×9 matrix.

EXERCISES FOR CHAPTER 7

For exercises 1 – 3:

(i) find the eigenvalues and eigenvectors;
(ii) determine whether or not it diagonalisable;
(iii) if it is diagonalisable, find an invertible matrix D and a diagonal matrix S such that

$$A = SDS^{-1};$$

(iv) if it is diagonalisable find a formula for A^n.

Exercise 1: $A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}$.

Exercise 2: $A = \begin{pmatrix} 3 & -2 & 4 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$.

Exercise 3: $A = \begin{pmatrix} 0 & -5 & 1 \\ 0 & 3 & -1 \\ 1 & 3 & 1 \end{pmatrix}$.

Exercise 4: Solve the recurrence system:

$$\begin{cases} a_{n+1} = 3a_n + 2b_n \\ b_{n+1} = 2a_n + 3b_n \\ a_0 = 1 \\ b_0 = 0 \end{cases}.$$

Exercise 5: Solve the recurrence equation:

$$\begin{cases} a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n \\ a_1 = 1 \\ a_2 = 2 \\ a_3 = 3 \end{cases}.$$

Exercise 6: For a certain type of insect 90% of those under week old die within the following week and 50% of those between 1 and 2 weeks old die within the following week. Very few survive beyond the first three weeks. They breed in the second and third weeks only. On average each female in the one to two week age group produces 30 offspring in that week. Half of these are female. For those in the two to three week age group each female produces 20 offspring in that week. Again half of these are female.

Let a_n be the number of thousands of females, aged 0 to 1 week, in week n. Let b_n be the number of thousands of females in the one to two week group in week n and let c_n be the number of thousands of females aged from 2 to 3 weeks in week n.

(i) How many weeks will it take before there are insects in all three age groups?

(ii) Let $\mathbf{v}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ Find a matrix M such that

$$\mathbf{v}_{n+1} = \mathrm{M}\mathbf{v}_n.$$

(iii) If 4000 females aged between 2 and 3 weeks are released into a certain area, and sufficient males to breed with them, find the population distribution after n weeks.

SOLUTIONS FOR CHAPTER 7

Exercise 1: (i) tr(A) = 3,

$$|A| = -10 - 18 = -28.$$

$\chi_A(\lambda) = \lambda^2 - 3\lambda - 28$

$\quad = (\lambda - 7)(\lambda + 4).$

So the eigenvalues are $\lambda = 7, -4$.

λ = 7: $A - 7I = \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$. So $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector.

λ = –4: $A + 4I = \begin{pmatrix} 9 & 6 \\ 3 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}$. So $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is an eigenvector.

(ii) Since we have two independent eigenvectors A is diagonalisable.

(iii) Let $S = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 7 & 0 \\ 0 & -4 \end{pmatrix}$.

Then $S^{-1} = \frac{1}{11}\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$.

$$SDS^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 21 & 8 \\ 7 & -12 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 55 & 66 \\ 33 & -22 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}.$$

(iv) $A^n = SDS^{n-1} = \frac{1}{11}\begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 7^n & 0 \\ 0 & (-4)^n \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$

$$= \frac{1}{11}\begin{pmatrix} 3\times 7^n & -2\times(-4)^n \\ 7^n & 3\times(-4)^n \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 9\times 7^n + 2\times(-4)^n & 6\times 7^n - 6\times(-4)^n \\ 3\times 7^n - 3\times(-4)^n & 2\times 7^n + 9\times(-4)^n \end{pmatrix}.$$

Exercise 2:

(i) $tr(A) = 3 + 2 - 1 = 4.$

$tr_2(A) = 4 + 1 + 0 = 5.$

$|A| = 3.0 + 2(-1) + 4.1 = 2.$

$\chi_A(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 2$

$= (\lambda - 1)^2(\lambda - 2).$

The eigenvalues are $\lambda = 1$ (twice), $\lambda = 2$.

λ = 1: $A - I = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ are independent eigenvectors (neither is a scalar multiple of the other).

λ = 2: $A - I = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 0 & -2 \\ -1 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 4 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{ so } \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \text{ is an eigenvector.}$$

(ii) Since we have 3 independent eigenvectors A is diagonalisable.

(iii) Let $S = \begin{pmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Then $|S| = -1 - (-2) - 2 = -1$.

$$S^{-1} = -\begin{pmatrix} -1 & -(1) & 1 \\ -(0) & 1 & -(1) \\ -2 & -(3) & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$\text{Then } SDS^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -4 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 4 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix} = A.$$

(iv) $A^n = SD^nS^{-1}$

$$= \begin{pmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2\times 2^n \\ 1 & 0 & 2^n \\ 0 & 1 & 2^n \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{n+1}-1 & 2-2^{n+1} & 2^{n+2}-4 \\ 1-2^n & 2^n & 2-2^{n+1} \\ 1-2^n & 2^n-1 & 3-2^{n+1} \end{pmatrix}.$$

Exercise 3:

(i) $tr(A) = 4$.

$tr_2(A) = 0 + (-1) + 6 = 5$.

$|A| = 0 + 5(1) + (-3) = 2$.

$\chi_A(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 2$ (as in exercise 2).

$= (\lambda - 1)^2(\lambda - 2)$.

So the eigenvalues are $\lambda = 1$ (twice), $\lambda = 2$.

$$\boldsymbol{\lambda = 1:}\ A - I = \begin{pmatrix} -1 & -5 & 1 \\ 0 & 2 & -1 \\ 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & -1 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 5 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boldsymbol{\lambda = 2:}\ A - 2I = \begin{pmatrix} -2 & -5 & 1 \\ 0 & 1 & -1 \\ 1 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ -2 & -5 & 1 \end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1 & 5 & -1\\ 0 & 1 & -1\\ 0 & 1 & -1\end{pmatrix}\rightarrow\begin{pmatrix}1 & 5 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}.$$

So $\begin{pmatrix}-4\\ 1\\ 1\end{pmatrix}$, and its scalar multiples, are the only eigenvectors for $\lambda = 1$.

(ii) Since we have only two independent eigenvectors A is not diagonalisable.

Exercise 4: Let $\mathbf{v}_n = \begin{pmatrix}a_n\\ b_n\end{pmatrix}$ and $A = \begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}$.

Then $\mathbf{v}_{n+1} = A\mathbf{v}_n$ and $\mathbf{v}_0 = \begin{pmatrix}10\\ 1\end{pmatrix}$.

tr(A) = 6 and |A| = 5.

$\chi_A(\lambda) = \lambda^2 - 6\lambda + 5$

$\quad = (\lambda - 1)(\lambda - 5)$.

The eigenvalues are $\lambda = 1, 5$.

λ = 1: $A - I = \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} \rightarrow \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$, so $\begin{pmatrix}1\\ -1\end{pmatrix}$ is an eigenvector.

λ = 5: $A - 5I = \begin{pmatrix}-2 & 2\\ 2 & -2\end{pmatrix} \rightarrow \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$, so $\begin{pmatrix}1\\ 1\end{pmatrix}$ is an eigenvector.

Let S = $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and D = $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$.

Then |S| = 2 and $S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

$$SD^nS^{-1}\mathbf{v}_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 5^n \\ -1 & 5^n \end{pmatrix}\begin{pmatrix} 9 \\ 11 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 9+11\times 5^n \\ -9+11\times 5^n \end{pmatrix}.$$

Hence $a_n = \dfrac{11\times 5^n + 9}{2}$ and $b_n = \dfrac{11\times 5^n - 9}{2}$.

Exercise 5: Let $\mathbf{v}_n = \begin{pmatrix} u_{n+2} \\ u_{n+1} \\ u_n \end{pmatrix}$ and A = $\begin{pmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then $\mathbf{v}_{n+1} = A\mathbf{v}_n$ and $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

tr(A) = 6, tr_2(A) = 11, |A| = 6.

$\therefore\ \chi_A(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$

$\qquad\qquad = (\lambda - 1)(\lambda - 2)(\lambda - 3)$

so the eigenvalues are λ = 1, 2, 3.

$$\boldsymbol{\lambda = 1:}\ A - I = \begin{pmatrix} 5 & -11 & 6 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -6 & 6 \\ 0 & 1 & -1 \end{pmatrix}$$

$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

$$\boldsymbol{\lambda = 2:}\ A - 2I = \begin{pmatrix} 4 & -11 & 6 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

so $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector.

$$\boldsymbol{\lambda = 3:}\ A - 3I = \begin{pmatrix} 3 & -11 & 6 \\ 1 & -3 & 0 \\ 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & -2 & 6 \\ 0 & 1 & -3 \end{pmatrix}$$

$\to \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 9 \\ 3 \\ 1 \end{pmatrix}$ is an eigenvector.

Let $S = \begin{pmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = .\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Then $|S| = -2$ and $S^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & 2 & -1 \\ 5 & -8 & 3 \\ -6 & 6 & -2 \end{pmatrix}^T$

$$= \frac{1}{2}\begin{pmatrix} -1 & 5 & -6 \\ 2 & -8 & 6 \\ -1 & 3 & -2 \end{pmatrix}.$$

$$A^n\mathbf{v}_0 = -\frac{1}{2}\begin{pmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}\begin{pmatrix} -1 & 5 & -6 \\ 2 & -8 & 6 \\ -1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 & 2^{n+2} & 3^{n+2} \\ 1 & 2^{n+1} & 3^{n+1} \\ 1 & 2^n & 3^n \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 & 2^{n+2} & 3^{n+2} \\ 1 & 2^{n+1} & 3^{n+1} \\ 1 & 2^n & 3^n \end{pmatrix}\begin{pmatrix} -1+2^{n+3}-3^{n+2} \\ -1+2^{n+2}-3^{n+1} \\ -1+2^{n+1}-3^n \end{pmatrix}.$$

Hence $u_n = \dfrac{3^n - 2^{n+1} + 1}{2}$.

Exercise 6: (i) The age distributions for females (in thousands) for the first few weeks are as follows.

weeks	**0**	**1**	**2**	**3**	**4**	**5**
0	0	40	0	60	20	90
1	0	0	4	0	6	2

2	4	0	0	2	0	3

So it is not until week 5 before there are insects in all age groups.

(ii) $$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 15 & 10 \\ 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}.$$

(iii) Let M be the above matrix.

$\mathrm{tr}(M) = 0$, $\mathrm{tr}_2(M) = -1.5$, $|M| = 0.5$.

$\chi_A(\lambda) = \lambda^3 - 1.5\lambda - 0.5.$

$\qquad = (\lambda - 1)(\lambda^2 + \lambda - 0.5).$

Hence the eigenvalues are 1, $\dfrac{-1 \pm \sqrt{3}}{2}$

$\qquad = 1, 0.366, -1.366.$

Since the eigenvalues are distinct M is diagonalisable and so each of a_n, b_n and c_n can be expressed in the form $A + 0.366^n B + (-1.366)^n C$.

Suppose $a_n = A + 0.366^n B + (-1.366)^n C$.

Since $a_0 = 0$, $a_1 = 40$ and $a_2 = 0$ we have:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0.366 & -1.366 \\ 1 & 0.366^2 & 1.366^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 40 \\ 0 \end{pmatrix}$$, that is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0.366 & -1.366 \\ 1 & 0.134 & 1.866 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 40 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0.366 & -1.366 & 40 \\ 1 & 0.134 & 1.866 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -0.634 & -2.366 & 40 \\ 0 & -0.866 & 0.866 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3.7319 & -63.0915 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3.7319 & -63.0915 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4.7319 & -63.0915 \end{array}\right)$$

$$\to. \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -13.3332 \end{array}\right)$$

So C = –13.3332, B = –13.3332, A = 26.6664.

Hence

$a_n = 26.6664 - 13.3332\times 0.366^n - 13.3332\times(-1.366)^n$.

Now $b_n = 0.1a_{n-1}$

$= 2.6666 - 1.333 \times 30.366^{n-1} - 1.3333 \times (-1.366)^{n-1}$.

Finally, $c_n = 0.5 b_{n-1}$

$= 1.3333 - 0.6666 \times 30.366^{n-2} - 0.6666 \times (-1.366)^{n-2}$.

0 weeks old	$\dfrac{80 - 40 \times 0.366^{n} - 40 \times (-1.366)^{n}}{3}$
1 week old	$\dfrac{8 - 4 \times 0.366^{n-1} - 4 \times (-1.366)^{n-1}}{3}$
2 weeks old	$\dfrac{4 - 2 \times 0.366^{n-1} - 2 \times (-1.366)^{n-1}}{3}$

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